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STACK SORTABLE PERMUTATIONS

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The class SS_n of stack sortable permutations is known to be in 1–1 correspondence with the set of n-noded binary trees. We use this correspondence to show that many properties of a binary tree are related to different types of monotonic subsequences in the corresponding permutation. Expressions are derived for the average length of these subsequences over the class SS_n. Also, it is shown that SS_n is directly related to special types of interval and comparability graphs.

1. Introduction

Given a permutation $\pi = \langle p_1, p_2, \ldots, p_n \rangle$ and an empty stack, the elements of π can be passed through the stack using two elementary operations coded "S" and "X". The operation "S" denotes "put the next element of π on top of stack" and "X" stands for "transfer the element on top of stack to the output". A sequence L of the above-mentioned operations is called a valid operation sequence (or simply an operation sequence) if and only if (1) all elements of π are transferred to the output and (2) the operation "X" is never specified when the stack is empty. Conditions (1) and (2) imply that an operation sequence must consist of $2n$ operations, n of each kind, where the number of "X" operations may never exceed the number of "S" operations when L is scanned from left to right. The sequence L is related in an obvious way to the classical ballot problem in which "S" is a vote for candidate A and "X" a vote for candidate B. L represents a ballot sequence where A and B each receive n votes, and B never has a majority during the counting process.

The class SS_n is studied in Knuth [3] and its relation to the classical ballot problem is shown in [4] and [7]. The correspondence between SS_n and the set of binary trees is used in [8] to generate and rank all "shapes" of n-noded binary trees. The cardinality of SS_n is

$$C_n = (n+1)^{-1}\binom{2n}{n}.$$

We denote by $L(\pi)$ the output permutation which results from passing π through a stack. For example if $\pi = \langle 1, 3, 2, 4 \rangle$ and $L = \langle S, X, S, X, S, S, X, X \rangle$ then $L(\pi) = \langle 1, 3, 4, 2 \rangle$. A permutation π is sortable with a stack if and only if there exists an operation sequence $\bar{L}$ such that $\bar{L}(\pi) = \langle 1, 2, \ldots, n\rangle$; it is realizable with a stack if and only if there exists an operation sequence $\bar{R}$ such that $\bar{R}(\langle 1, 2, \ldots, n\rangle) = \pi$.

Given a permutation π, let $\bar{L}$ be the sequence of operations which sorts π with a stack. Scanning $\bar{L}$ from left to right, we call each sequence of consecutive "S" operations an *S-group* and each sequence of "X" operations an *X-group*. Clearly, the number of X-groups is equal to the number of S-groups, two S-groups are separated by an X-group and vice versa. The *S-specification* and the *X-specification* of $\bar{L}$ are vectors $\langle s_1, s_2, \ldots, s_l\rangle$ and $\langle x_1, x_2, \ldots, x_l\rangle$ respectively, where s_i denotes the size of the ith S-group and x_i the size of the ith X-group, for $1 \leq i \leq l$.

We denote by SS_n the class of permutations of order n which are sortable with a stack, and by SR_n the class of permutations of the same order which are relaizable with a stack. These two classes are related as follows:

$$\pi \in \mathrm{SS}_n \quad \text{if and only if} \quad \pi^{-1} \in \mathrm{SR}_n. \tag{1}$$

The class SR_n is characterized by Knuth [3, Ex. 2.2.1,5] by the following theorem:

Theorem 1. *The permutation $\pi = \langle p_1, p_2, \ldots, p_n\rangle$ is a member of* SR_n *if and only if it does not contain a subsequence*

$$\langle p_i, p_j, p_k\rangle \quad \textit{such that} \quad p_i > p_k > p_j. \tag{2}$$

From this theorem and the relation (1) we obtain a characterization of SS_n as follows:

Theorem 2. $\pi \in \mathrm{SS}_n$ *if and only if it does not contain a subsequence*

$$\langle p_i, p_j, p_k\rangle \quad \textit{such that} \quad p_j > p_i > p_k. \tag{3}$$

For a node j in a binary tree T, we denote by $L_T(j)$ and $R_T(j)$ the left and right subtrees of j, respectively.

The symmetric traversal of a binary tree with root x is defined recrusively as follows: if the tree is empty do nothing, else traverse $L_T(x)$ then visit x and then traverse $R_T(x)$. A non-recursive algorithm for this traversal which uses a stack is given in [3, 2.3.1]; we shall assume the reader is familiar with this algorithm.

A permutation π can be mapped into a labelled binary tree T_π using the following well-known construction.

Construction *T*. Given $\pi = \langle p_1, p_2, \ldots p_n\rangle$ and an empty tree T, assign p_1 to the root of the tree; for each p_k, $k = 2, 3, \ldots, n$ apply the rule

> "if p_k is inserted into a non-empty subtree rooted by p_i, it is inserted into $L_T(p_i)$ if $p_k < p_i$ otherwise p_k is inserted into $R_T(p_i)$"

until an empty subtree is reached and then a root labeled p_k is created for that subtree. (See illustration near Fig. 1 for two permutations and their corresponding trees.)

Construction T establishes a 1–1 correspondence between the set SS_n and the set of n-noded binary trees [3; Ex. 2.3.1,6]. Given a labeled tree T_π, the corresponding member of SS_n can be obtained by reading the labels of T_π in preorder (root, left subtree and right subtree) [3; 2.3.1].

In this paper we study in detail some of the combinatorial properties of the class SS_n. In Sections 2 and 3 expressions are derived for the expected length of some types of monotonic subsequences and the average number of inversions. In the last section the permutation graph associated with $\pi \in SS_n$ is shown to be an interval graph of a special type.

2. Monotonic subsequences in SS_n and their relation to binary trees

Let $\pi = \langle p_1, p_2, \ldots, p_n\rangle$ be a permutation on the set $N = \{1, 2, \ldots, n\}$. A *descending subsequence* of length k in π satisfies

$$p_{i_1} > p_{i_2} > \cdots > p_{i_k} \quad \text{and} \quad i_1 < i_2 < \cdots < i_k.$$

A descending subsequence is maximal in π if no element of π can be added to it without violating its monotonicity. A *longest descending subsequence* in π (LDS) contains the maximum number of elements among all descending subsequences in π. We get the corresponding definitions for ascending subsequences by replacing ">" with "<" in the above, with LAS standing for *longest ascending subsequence*. For $j \in N$, denote by $R_\pi(j)$ the set of elements to the right of j in π, and by $L_\pi(j)$ the set of elements to the left of j in π. Two elements p_i and p_j form an *inversion* in π if $(p_i - p_j)(i-j) < 0$.

A *descending run* in π is a sequence of successive elements $p_i, p_{i+1}, \ldots, p_{i+k}$ such that

$$p_{i-1} < p_i > p_{i+1} > \cdots > p_{i+k} < p_{i+k+1}.$$

(We assume that $p_1 > p_0$ and $p_n < p_{n+1}$.)

The *inversion table* of π is a vector $\langle b_1, b_2, \ldots, b_n\rangle$ such that b_i counts the number of elements in $R_\pi(i)$ which are smaller than i for $1 \leq i \leq n$. It is well known that an inversion table uniquely determines its corresponding permutation.

Example. Let $\pi = \langle 3, 6, 4, 5, 2, 1\rangle$. Then $\langle 3, 2, 1\rangle$ is a maximal descending subsequence in π, $\langle 6, 4, 2, 1\rangle$ and $\langle 3, 4, 5\rangle$ are an LDS and an LAS respectively in π, $R_\pi(4) = \langle 5, 2, 1\rangle$ and $L_\pi(6) = \langle 3\rangle$. The inversion-table of π is $\langle 0, 1, 2, 2, 2, 4\rangle$.

Theorem 3. *The length of an LDS in $\pi \in SS_n$ is equal to the depth of stack which is needed to traverse T_π in symmetric order.*

Proof. We observe that the sequence of insertions and removals from the stack made during the symmetric traversal of T_π is equivalent to the sequence of operations required to sort π with a stack.

Let $D=\langle d_{i_1}, d_{i_2}, \ldots, d_{i_l}\rangle$ be an LDS in π. While sorting π, no member of D can leave the stack before d_{i_l}, so the stack must have at least l entries.

Conversely, assume that the stack contains m elements during the sorting process and $m>l$. Let $B=\langle b_{i_1}, b_{i_2}, \ldots, b_{i_m}\rangle$ be the elements in the stack, then B must be a descending subsequence in π, which contradicts the definition of D. $\square$

Corollary. *The expected length of an* LDS *in a random permutation of* SS_n *(each permutation chosen with probability* C_n^{-1}*) is asymptotically*

$$\sqrt{\pi n}-1.5+\frac{11}{24}\sqrt{\frac{\pi}{n}}+O(n^{-\frac{3}{2}}). \tag{4}$$

Proof. This follows from Knuth's result on the average depth of stack [3, Ex. 2.3.1, 11] $\square$

Remark. The problem of finding the expected length of an LDS (or an LAS) in a random permutation (where each permutation has probability $(n!)^{-1}$) is still unsolved analytically. Experimental results show good agreement with $2\sqrt{n}$ [2].

We need the following definitions to prove the corresponding result on the LAS.

A *composition* of a whole number n into m parts (also called *ordered partition*) is a vector $C=\langle c_1, c_2, \ldots, c_m\rangle$ such that $c_i>0$ for $1\leq i\leq m$ and $\sum_{i=1}^m c_i=n$. A composition C of n can be represented as a zig-zag graph, this graph contains m rows with c_i dots in the ith row, for $i>1$ the first dot in the ith row is written under the last dot in row $i-1$. Given a composition C, we obtain its *conjugate* composition $\bar{C}=\langle\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{n+1-m}\rangle$ where $\bar{c}_i$ is equal to the number of dots in the ith column (from left) of the zig-zag graph of C for $1\leq i\leq n+1-m$. For example let $C=\langle 3, 2, 4, 1\rangle$ be a composition of the integer 10. The zig-zag graph of C is

$$\begin{array}{ccccccc}
\cdot & \cdot & \cdot & & & & \\
 & & \cdot & \cdot & & & \\
 & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \cdot
\end{array}$$

therefore $\bar{C}=\langle 1, 1, 2, 2, 1, 1, 2\rangle$.

Let π and π_{RF} be two members of SS_n (not necessarily distinct) such that their corresponding trees T_π and $T_{\pi_{\mathrm{RF}}}$ are reflections of each other about the vertical axis.

Lemma 1. *Let* $X=\langle x_1, x_2, \ldots, x_k\rangle$ *and* $X_{\mathrm{RF}}=\langle x'_1, x'_2, \ldots, x'_m\rangle$ *be the* X-

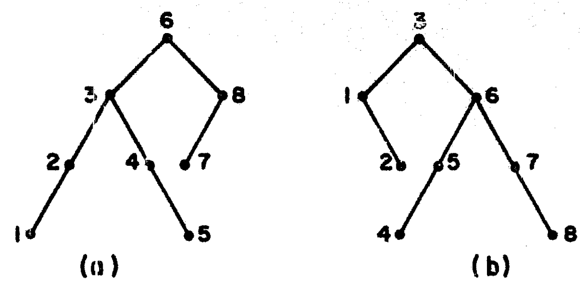

Fig. 1.

specifications of $\bar{L}$ *and* $\bar{L}_{RF}$ *respectively. Then the vectors* $X^R = \langle x_k, x_{k-1}, \ldots, x_1 \rangle$ *(the reverse of X) and* X_{RF} *are conjugate compositions of n.*

Illustration. Consider the permutations $\pi = \langle 6, 3, 2, 1, 4, 5, 8, 7 \rangle$ and $\pi_{RF} = \langle 3, 1, 2, 6, 5, 4, 7, 8 \rangle$. The corresponding binary trees are shown in Figs. 1 (a) and (b) respectively.

The X-specification of $\bar{L}$ is $X = \langle 3, 1, 2, 2 \rangle$ and $X^R = \langle 2, 2, 1, 3 \rangle$. The zig-zag graph of X^R is

and therefore its conjugate is $\bar{X}^R = \langle 1, 2, 3, 1, 1 \rangle$, we then have $\bar{X}^R = X_{RF}$.

Proof. A binary tree is traversed in reverse symmetric order if a root and its two subtrees are visited in the order (1) right subtree (2) root (3) left subtree. We observe that the operations which are required in order to traverse T_π in reverse symmetric order are equivalent to those necessary for traversing $T_{\pi_{RF}}$ in symmetric order. Therefore $\bar{L}$ and $\bar{L}_{RF}$ specify the stack operations for traversing T_π in symmetric and reverse symmetric order respectively. For two consecutive labels i and $i-1$ we can have (a) $i \in R_{T_\pi}(i-1)$ or (b) $i-1 \in L_{T_\pi}(i)$. While traversing T_π in symmetric order, (a) implies that i must be stacked after $i-1$ is written on output and therefore $X(i-1)$ and $X(i)$ are in different X-groups, (b) implies that i is present in the stack when $i-1$ is written, hence $X(i)$ and $X(i-1)$ are in the same X-group. It is easy to see that in the reverse symmetric order traversal of T_π we have exactly the converse, i.e. the labels i and $i-1$ are written on output by the same X-group in $\bar{L}_{RF}$ in case (a) and by different X-groups in case (b).

We can represent X as a zig-zag graph in which the ith row contains the elements written by the ith X-group in $\bar{L}$. By the above argument, it follows that the ith X-group in $\bar{L}_{RF}$ will write out elements of the ith column in this graph, where counting starts from the rightmost column. For example in the above illustration the graph is

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123
  4
  56
   78
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and the X-groups of $\bar{L}_{RF}$ write out $\langle 8\rangle$, $\langle 7, 6\rangle$, $\langle 5, 4, 3\rangle$, $\langle 2\rangle$, $\langle 1\rangle$, where brackets enclose elements of the same X-group. Therefore X^R and X_{RF} are conjugate compositions and $k = n+1-m$. □

Lemma 2. *The length of an* LAS *in* $\pi \in \mathrm{SS}_n$ *is equal to the number of components in the S-specification (X-specification) of its sorting sequence.*

Proof. Let $\bar{L}$ be a sorting sequence for π with S-specification $\langle s_1, s_2, \ldots, s_l\rangle$. Then clearly π must have exactly l descending runs where the size of the ith run is s_i. Let an LAS in π be of length k. Then $k \leq l$ since no two elements in an LAS are in the same descending run. To show that $l \leq k$, we construct a sequence $D = \langle d_1, d_2, \ldots, d_l\rangle$ where d_i is the last element in the ith descending run in π.

Note that the d_i are the first elements output in the ith X-group; therefore they must be an ascending subsequence. □

Theorem 4. *The expected length of an* LAS *in a random permutation of* SS_n *is* $\frac{1}{2}(n+1)$.

Proof. We define a mapping $\mathrm{RF}: \mathrm{SS}_n \to \mathrm{SS}_n$ such that $\pi \in \mathrm{SS}_n$ is mapped into π_{RF} by RF. Suppose that the length of the LAS in π is equal to k. By Lemma 2 this is also the number of components in the S-specification and X-specification of the sorting sequence $\bar{L}$. From Lemma 1, the length of the LAS in π_{RF} is $n+1-k$. Since RF is a one-to-one correspondence our result follows. □

Corollary. *The average number of nodes with no left son in a random n-noded binary tree is* $\frac{1}{2}(n+1)$.

Proof. Consider the sequence of stack operations during a symmetric traversal of a binary tree. There is a change from "S" to "X" in this sequence exactly when a node with no left son is encountered. The result now follows from Lemma 2 and Theorem 4. □

This result is proved in Knuth [3] in another way.

Another subsequence which has been studied in permutations is the sequence of left to right maxima; an element j is a left to right maximum if there is no element x on its left such that $x > j$. This sequence is also called the distinguished subsequence by Brock & Baer [1]. For example, the distinguished subsequence of

$\langle 1,3,2,5,4,6\rangle$ is $\langle 1,3,5,6\rangle$. It is shown by Knuth [3, 4] and in [1] that the expected length of this subsequence in a random permutation on $\{1,2,\ldots,n\}$ is H_n (the nth harmonic number). The next theorem gives the corresponding result for a random permutation in SS_n.

Theorem 5. *The expected length of the distinguished subsequence in a random permutation of* SS_n *is* $3-6/(n+2)$.

Proof. Given $\pi=\langle p_1,p_2,\ldots,p_n\rangle\in SS_n$ let $\langle p_{i_1},p_{i_2},\ldots,p_{i_k}\rangle$ be the distinguished subsequence in π. We can form $k+1$ permutations $\pi_1,\pi_2,\ldots,\pi_{k+1}$ of length $n+1$ from π by inserting the number $n+1$ in each of the positions immediately to the left of p_{i_j} in π for $1\leq j\leq k$ or placing $n+1$ as the last element in π. For example, if $\pi=\langle 1,3,2,4\rangle$ then $\langle 5,1,3,2,4\rangle$ $\langle 1,5,3,2,4\rangle$ $\langle 1,3,2,5,4\rangle$ and $\langle 1,3,2,4,5\rangle$ are formed in this way. We now show that $\pi_i\in SS_{n+1}$ for $1\leq i\leq k+1$. If not, then some π_j, $2\leq j\leq k$, must contain a forbidden subsequence of the form $\langle p_i, n+1, p_l\rangle$ where $p_i>p_l$

$$i<l. \tag{5}$$

But then $\langle p_i, p_{i_j}, p_l\rangle$, where p_{i_j} is the distinguished element immediately to the right of $n+1$, is also a forbidden subsequence, which contradicts $\pi\in SS_n$. It is easy to see that inserting $n+1$ in any other position of π will create a permutation π' such that $\pi'\notin SS_{n+1}$. On the other hand all of the members of SS_{n+1} can be generated from the members of SS_n in this way. Let a_π be the length of the distinguished subsequence in π. Then

$$|SS_{n+1}|=C_{n+1}=\sum_{\pi\in SS_n}(a_\pi+1), \tag{6}$$

$$C_{n+1}=\sum a_\pi+C_n, \tag{7}$$

$$\frac{\sum a_\pi}{C_n}=\frac{C_{n+1}}{C_n}-1=\frac{\dfrac{1}{n+2}\dbinom{2n+2}{n+1}}{\dfrac{1}{n+1}\dbinom{2n}{n}}-1, \tag{8}$$

which gives

$$\frac{\sum a_\pi}{C_n}=3-\frac{6}{n+2}. \quad \square \tag{9}$$

Remark. This result is directly related to random walks on binary trees as described by Munro [5]. Such a walk starts at the root and takes a right or left branch with probability p and $1-p$ respectively. The walk is stopped when a chosen branch does not exist in the tree. Munro proves that when $p=\frac{1}{2}$ the average length of a random walk on a random binary tree is $2-6/(n+2)$. It is easy to show, using symmetry arguments, that this result holds for any probability

p. Given $\pi \in \mathrm{SS}_n$, the elements of the distinguished subsequence in π form the rightmost path in T_π. Therefore the result of Theorem 5 is equivalent to the case of random walk with $p=1$.

Corollary. *The expected length of the first descending run in a random permutation of* SS_n *is* $3-6/(n+2)$.

Proof. By symmetry, the average length of the leftmost path over all n-noded binary trees is also $3-6/(n+2)$. This path in T_π is formed by the members of the first descending run in π, since under Construction T this path is completed before any other part of the tree is constructed. □

4. The average number of inversions in SS_n

Lemma 3. *Let* $\langle b_1, b_2, \ldots, b_n\rangle$ *be the inversion-table of* $\pi \in \mathrm{SS}_n$, *then for the node labelled* k *in* T_π $|L_{T_\pi}(k)| = b_k$.

Proof. Clearly an element j is inserted into $L_{T_\pi}(k)$ by Construction T if and only if $j<k$ and $j \in R_\pi(k)$ and therefore $b_k = |L_{T_\pi}(k)|$.

Theorem 6. *The average number of inversions* $\bar{I}_n$ *in a random permutation of* SS_n *is equal to half the average internal path length of a binary tree of* n *nodes, that is*

$$\bar{I}_n = \frac{1}{2}\left(\frac{4^n}{C_n} - 3n - 1\right). \tag{10}$$

Proof. *Let* $i(\pi)$ denote the number of inversions in a permutation π and $\mathrm{int}(T)$ the internal path length of the tree T. The sum of sizes of all proper subtrees in a binary tree (or any other tree) is equal to $\mathrm{int}(T)$. This follows from the fact that in a tree T, the distance of vertex i from the root is equal to the number of subtrees in which i is contained.

Let $\langle b_1, b_2, \ldots, b_n\rangle$ be the inversion table of a permutation $\pi \in \mathrm{SS}_n$, then by definition

$$\sum_{i=1}^{n} b_i = i(\pi). \tag{11}$$

By Lemma 3, $i(\pi)$ is the sum of sizes of all left subtrees in T_π. Hence, by the symmetry of left and right subtrees

$$\sum_{\pi \in \mathrm{SS}_n} \mathrm{int}(T_\pi) = 2 \sum_{\pi \in \mathrm{SS}_n} i(\pi). \tag{12}$$

The value of the left member of (12) is given in [3, Ex. 2.3.4.5,4] as

$$\sum_{\pi \in \mathrm{SS}_n} \mathrm{int}(T_\pi) = 4^n - (3n+1)C_n, \tag{13}$$

from which (10) follows. □

It is interesting to note that on the average a random permutation of SS_n contains $O(n^{1.5})$ inversions, whereas the corresponding value for a random permutation on $\{1, 2, \ldots, n\}$ is $O(n^2)$.

5. Graphs associated with SS_n

We give some definitions and notations from graph theory which are required in this section.

A *graph* $G(V, E)$, consists of a vertex set V and an edge set E, such that each edge in E is associated with two vertices in V called its *end points*. We consider here only graphs which have no two edges with the same two end points (parallel edges), and no edge for which its two end points are the same (self loop). We denote by v_i——v_j the existence of an edge between v_i and v_j otherwise v_i—/—v_j.

A direction can be assigned to the edge v_i——v_j, this is denoted by $v_i \rightarrow v_j$. A digraph is *transitive* if for $v_i, v_j, v_k \in V$, the existence of $v_i \rightarrow v_j$ and $v_j \rightarrow v_k$ implies $v_i \rightarrow v_k$. A graph G is a *comparability graph* if it is possible to orient all its edges such that its directed image is transitive. Thus, a non-directed graph $G(V, E)$ is a comparability graph if and only if there exists a partial ordering $<$ of V such that an edge connects two vertices $x, y \in V$ if and only if they are comparable, i.e. $x < y$ or $y < x$.

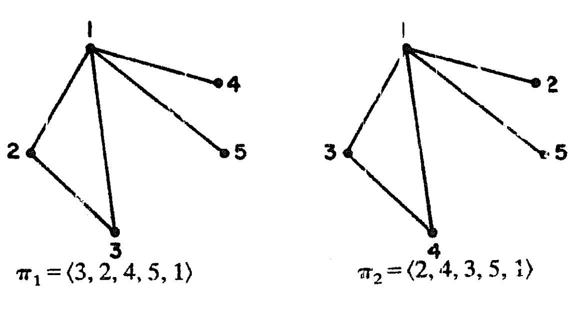
Let $G(N)$ be a graph which has its vertices labeled by the set $N = \{1, 2, \ldots, n\}$. Then $G(N)$ has a *defining permutation* with respect to its labeling if there is a permutation π on N such that: i——j (vertices are called by their labels) if and only if i and j form an inversion in π.

Clearly, if we orient the edges of such a graph from the vertex with the smaller label to the one with the bigger label we obtain a transitive orientation.

A graph G is a *permutation graph* if at least one of the possible labelings of its vertices with N gives rise to a defining permutation.

Example. A permutation graph G, with two labelings and their respective defining permutations, is shown in Fig. 2.

The next theorem of [6] demonstrates the connection between permutation graphs and comparability graphs.



$\pi_1 = \langle 3, 2, 4, 5, 1 \rangle$ $\qquad$ $\pi_2 = \langle 2, 4, 3, 5, 1 \rangle$

Fig. 2.

Theorem 7. *A graph G is a permutation graph if and only if both G and G^C are comparability graphs. (G^C is the complement of G.)*

A graph G with vertex set $V(|V|=n)$ is an *interval graph* if there exists a family of intervals $I=(I_1, I_2, \ldots, I_n)$ on the line such that $v_i \in V$ corresponds to an interval I_i, and v_i——v_j if and only if $I_i \cap I_j \neq \emptyset$. A *nested interval graph* is an interval graph which has a representing family I such that for each pair of intervals I_i and I_j, if $I_i \cap I_j \neq \emptyset$ then either $I_i \subset I_j$ or $I_j \subset I_i$ holds.

Theorem 8. *The following conditions are equivalent:*

(1) *G is a permutation graph, with a defining permutation $\pi \in \mathrm{SS}_n$.*

(2) *G is a nested interval graph.*

Proof. (1)→(2); Consider the sorting sequence of π, where a line is drawn from each S operation to the corresponding X operation which removes from the stack the element stacked by S. For example, for $\pi = \langle 3, 1, 2\rangle$ the following sorting sequence and lines are drawn:

$$S\ S\ X\ S\ X\ X.$$

Let I_i be the line drawn between the S and X which stack and unstack i in π. For a pair of intervals I_i and I_j assume that I_i has its left end to the left of I_j $(i \in L_\pi(j))$. Then two cases are possible:

(a) $i<j$, i leaves the stack before j is stacked and $I_i \cap I_j = \emptyset$;

(b) $i>j$, i leaves the stack only after j is unstacked and $I_i \supset I_j$. In the permutation graph G labeled with π, vertices labeled i and j are adjacent only in case (b) where i and j form an inversion in π; hence G is a nested interval graph. (2)→(1); Conversely, let I be a family of n intervals which represents the nested interval graph G. Then, I can be mapped into a sequence of S's and X's by reversing the above procedure. By reading this sequence of S's and X's from left to right we obtain a sorting sequence of some $\pi \in \mathrm{SS}_n$ and π is a defining permutation for G. □

A partially ordered set is called a tree (Wolk [9]) if for any pair of incomparable elements x, y there is no z such that $z<x$ and $z<y$. For a permutation π we define the relation $<$ as follows: for $x, y \in \pi$, $x<y$ if and only if x and y form an inversion in π and x is smaller than y. Using these definitions we obtain the following corollary.

Corollary. *A graph G is a nested interval graph if it is a comparability graph of a tree.*

Proof. A permutation $\pi \in \mathrm{SS}_n$ is a tree with respect to the relation $<$ since it does not contain a forbidden subsequence (3). Therefore the permutation graph which corresponds to π is a comparability graph of a tree. □

The converse of this corollary is also true and can be proved by using Wolk's algorithm which finds a partially ordered set which is a tree from a given comparability graph of a tree.

In the next theorem we characterize permutation graphs whose defining permutation is both sortable and recognizable with a stack.

Theorem 9. *The following conditions are equivalent*

(a) *G is a union of vertex disjoint complete subgraphs.*

(b) *G is a permutation graph with a defining permutation $\pi \in SS_n \cap SR_n$.*

Proof. Assume G has property (a) and let $K^1, K^2, \ldots, K^l$ be its complete subgraphs. We can number the vertices of G from 1 to n such that the vertices of K^i $(1 \leq i \leq l)$ are consecutive integers and the vertices of K^i are smaller than those of K^j for $i<j$. Let S^i be the sequence of integers assigned to the vertices of K^i written in decreasing order. Then the permutation $\pi = \langle S^1, S^2, \ldots, S^l \rangle$ obtained by concatenating the subsequences S^i, is a defining permutation for G since elements $x, y \in \pi$ form an inversion if and only if $x, y \in S^i$ for some i. Also $\pi \in SS_n \cap SR_n$ because it cannot contain any subsequences of type (2) or (3).

(b)$\rightarrow$(a); We observe that $\pi \in SS_n \cap SR_n$ has the property that a pair of elements x, y with x smaller than y are an inversion in π iff x, y appear in consecutive positions in π (y preceding x) and $y = x+1$. In any other case a forbidden subsequence (2) or (3) will be formed. We can therefore partition π from left to right into disjoint subsequences $S^1, S^2, \ldots, S^l$ where all elements of S^i are consecutive integers written in decreasing order and each element of S^i is smaller than the elements of S^j for $i<j$. Therefore a permutation graph with a defining permutation $\pi \in SS_n \cap SR_n$ must have property (a).

Corollary. *The number of permutations in $SS_n \cap SR_n$ is 2^{n-1}.*

Proof. The number of ways to partition the integers 1 to n into k disjoint subsequences S^i as defined above is $\binom{n-1}{k-1}$ and therefore

$$|SS_n \cap SR_n| = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}. \quad \square$$

Conclusions

In this paper we studied some of the combinatorial properties of members of SS_n, and the relations of these properties to the corresponding binary tree and other graphs. It was observed that members of SS_n tend to be more "ordered" than ordinary permutations in the sense that on the average they contain less inversions, longer maximum ascending subsequences and shorter maximum descending subsequences.

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